Monday Night Calculus, February 21, 2022

1. If $g(x) = \displaystyle\int_1^{x^2} \frac{3t}{t^3+1}\,dt$, then what is the value of $g'(2)$? (Lindsay Schiller Dunkin)

Solution

$$\frac{d}{dx}\left[\int_1^{x^2} \frac{3t}{t^3+1}\,dt\right] = \frac{d}{dx}\left[\int_1^{u} \frac{3t}{t^3+1}\,dt\right] \qquad \text{Let } u = x^2.$$

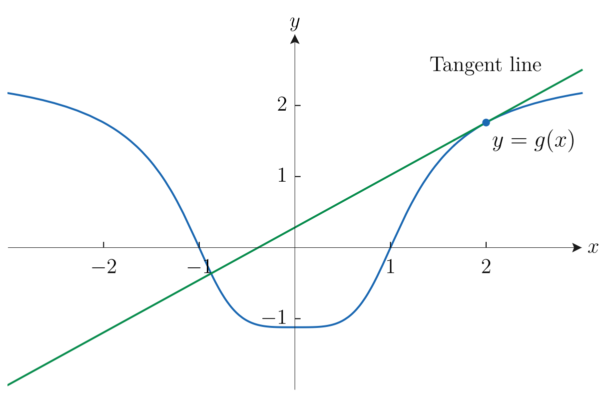
$$= \frac{d}{du}\left[\int_1^{u} \frac{3t}{t^3+1}\,dt\right]\cdot\frac{du}{dx} \qquad \text{Chain Rule.}$$

$$= \frac{3u}{u^3+1}\cdot\frac{du}{dx} \qquad \text{FTC, part 1.}$$

$$= \frac{3x^2}{x^6+1}\cdot 2x = \frac{6x^3}{x^6+1} \qquad \text{Use } u = x^2\text{; simplify.}$$

$$g'(x) = \frac{6x^3}{x^6+1}$$

$$g'(2) = \frac{6\cdot 2^3}{2^6+1} = \frac{48}{65}$$



2. Solve the initial value problem $\dfrac{dy}{dx} = \dfrac{y+2}{x+1}$, $y(0) = 1$. (Michelle Morman Owen)

Solution

$$\frac{dy}{dx} = \frac{y+2}{x+1}$$ Differential Equation.

$$\frac{dy}{y+2} = \frac{dx}{x+1}$$ Separate variables.

$$\ln|y+2| = \ln|x+1| + C$$ Antiderivatives.

$$\ln|1+2| = \ln|0+1| + C \quad \Rightarrow \quad C = \ln 3$$ Use initial condition.

$$\ln(y+2) = \ln(x+1) + \ln 3$$ Use $C = \ln 3$; branches.

$$y + 2 = 3(x+1) \quad \Rightarrow \quad y = 3x + 1$$ Solve for y.

Domain: $x > -1$

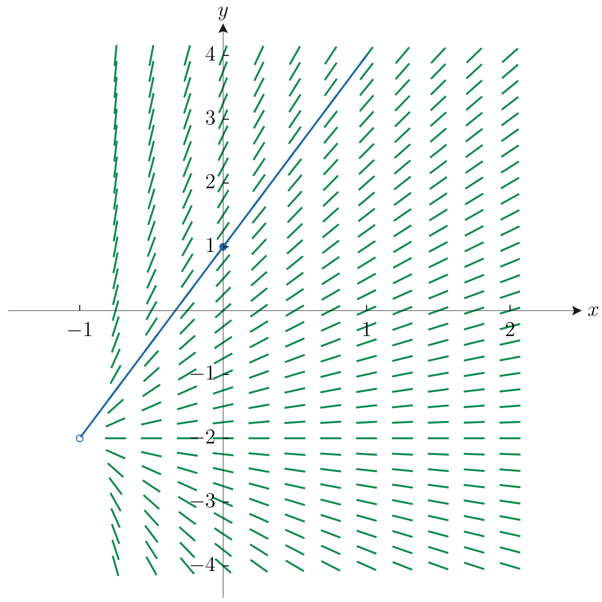
The Domain of Solutions to Differential Equations, Larry Riddle

The domain of a particular solution to a differential equation is the largest open interval containing the initial value on which the solution satisfies the differential equation. Some textbook authors call the domain of a solution the *interval of definition* of the solution or the *maximum interval of existence*.

Why are domains (open) intervals?

We want the differentiability of the solution to imply the intuitive concept of continuity that we often teach pre-calculus students: a function is continuous if you can draw its graph without lifting your pencil.

Stephen Saperstone: ... an explicit solution of an ODE, in order to be meaningful and useful, must be defined on an interval.



y
x
4
3
2
1
−1
−2
−3
−4
−1
1
2

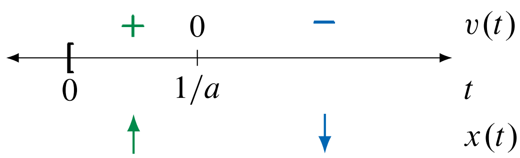
3. A particle moves along the x-axis so that at any time $t \geq 0$ its position is given by $x(t) = te^{-at}$, where a is a positive constant. At what time t is the particle's position farthest to the right? (Nicholas Frederick Bennett)

Solution

$$v(t) = x'(t) = 1 \cdot e^{-at} + t \cdot (-a)e^{-at} = e^{-at}(1 - at)$$

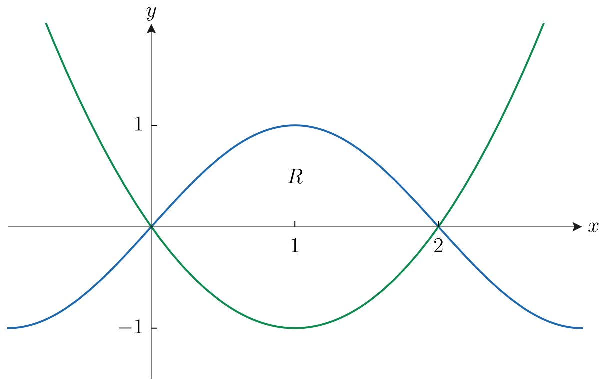
$$v(t) = 0 : 1 - at = 0 \quad \Rightarrow \quad t = \frac{1}{a}$$

$v(t)$ DNE : none



The particle is farthest to the right when $t = \dfrac{1}{a}$.

4. Let R be the region bounded by the graphs of $y = \sin\left(\frac{\pi}{2}x\right)$ and $y = x^2 - 2x$ between $x = 0$ and $x = 2$ (as shown in the figure).



(a) Find the area of the region R.

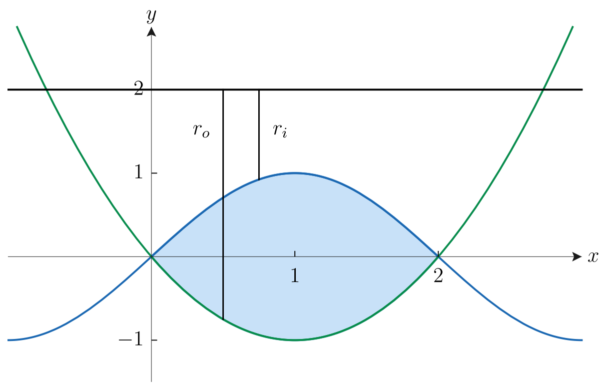
$$A = \int_0^2 \left[\sin\left(\frac{\pi}{2}x\right) - (x^2 - 2x)\right] dx = 2\int_0^1 \left[\sin\left(\frac{\pi}{2}x\right) - (x^2 - 2x)\right] dx$$

$$= 2\left[-\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right) - \frac{x^3}{3} + x^2\right]_0^1$$

$$= 2\left[\left(-\frac{2}{\pi}\cos\left(\frac{\pi}{2}\right) - \frac{1}{3} + 1\right) - \left(-\frac{2}{\pi}\cos(0) - 0 + 0\right)\right]$$

$$= 2\left[\frac{2}{3} + \frac{2}{\pi}\right] = \frac{4}{3} + \frac{4}{\pi}$$

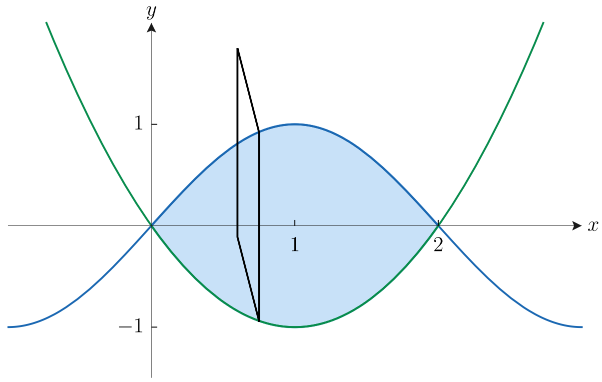
(b) Find the volume of the solid obtained by rotating the region R about the line $y = 2$.



$$V = 2\pi \int_0^1 \left[\left(2 - (x^2 - 2x)\right)^2 - \left(2 - \sin\left(\frac{\pi}{2}x\right)\right)^2 \right] dx$$

$$= \cdots = 2\pi \left(\frac{27}{10} + \frac{8}{\pi} \right) = \frac{27\pi}{5} + 16$$

(c) The base of a solid S is the region R. For this solid, cross-sections perpendicular to the x-axis are squares. Find the volume of this solid.

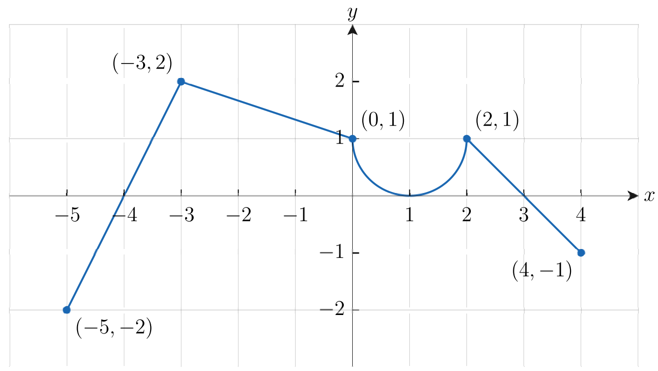


$$A(x) = s^2 = \left[\sin\left(\frac{\pi}{2}x\right) - (x^2 - 2x)\right]^2$$

$$V = 2\int_0^1 \left[\sin\left(\frac{\pi}{2}x\right) - (x^2 - 2x)\right]^2\, dx$$

$$= \cdots = 2\left(\frac{31}{30} + \frac{32}{\pi^3}\right)$$

5. The graph of the function f shown in the figure consists of a semicircle and three line segments. Let g be the function given by $g(x) = \int_{-3}^{x} f(t)\,dt$. (Ben Gazy)



Graph of f

(a) Find $g(0)$ and $g'(0)$.

$$g(0) = \int_{-3}^{0} f(t)\,dt = 3 + \frac{1}{2} \cdot 3 \cdot 1 = \frac{9}{2}$$

$$g'(0) = f(0) = 1$$

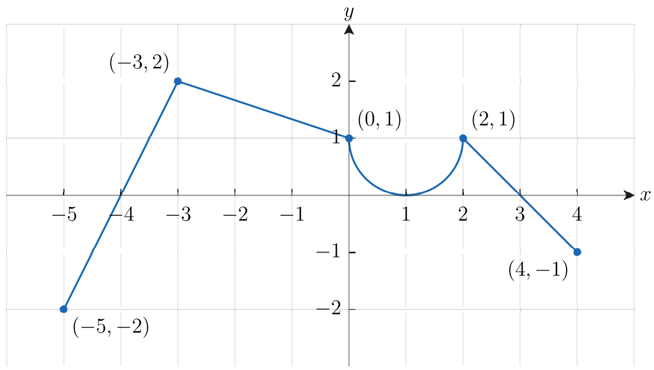
(b) Find all the values of x in the open interval $(-5, 4)$ at which g attains a relative maximum. Justify your answer.

$$g'(x) = f(x) = 0 : x = -4,\ 1,\ 3$$

$$g'(x) = f(x) \text{ DNE : none}$$

g attains a relative maximum at $x = 3$ because $g'(x) = f(x)$ changes from positive to negative at $x = 3$.

(c) Find the absolute minimum value of g on the interval $[-5, 4]$. Justify your answer.



Graph of f

Use the Table of Values Method, or Candidates Test.

Check the value of $g(x)$ at each endpoint of the closed interval and at each critical value in the open interval.

x	$g(x)$
-5	0
-4	-1
1	$\frac{11}{2} - \frac{\pi}{4}$
3	$7 - \frac{\pi}{2}$
4	$\frac{13}{2} - \frac{\pi}{2}$

$$g(-5) = \int_{-3}^{-5} f(t)\,dt = -\int_{-5}^{-3} f(t)\,dt = -\left(-\frac{1}{2}\cdot 1\cdot 2 + \frac{1}{2}\cdot 1\cdot 2\right) = 0$$

$$g(-4) = \int_{-3}^{-4} f(t)\,dt = -\int_{-4}^{-3} f(t)\,dt = -\frac{1}{2}\cdot 1\cdot 2 = -1$$

$$g(1) = \int_{-3}^{1} f(t)\,dt = \int_{-3}^{0} f(t)\,dt + \int_{0}^{1} f(t)\,dt$$

$$= \left(3 + \frac{1}{2}\cdot 3\cdot 1\right) + \left(1 - \frac{1}{4}\cdot \pi\cdot 1^2\right) = \frac{9}{2} + 1 - \frac{\pi}{4} = \frac{11}{2} - \frac{\pi}{4}$$

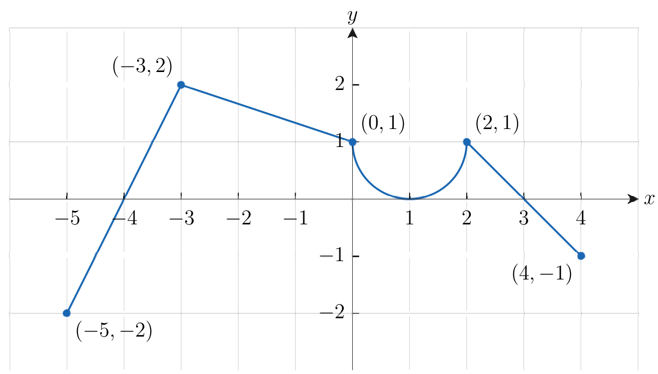
$$g(3) = \int_{-3}^{3} f(t)\,dt = \int_{-3}^{1} f(t)\,dt + \int_{1}^{2} f(t)\,dt + \int_{2}^{3} f(t)\,dt$$

$$= \frac{11}{2} - \frac{\pi}{4} + \left(1 - \frac{\pi}{4}\right) + \frac{1}{2}\cdot 1\cdot 1 = 7 - \frac{\pi}{2}$$

$$g(4) = \frac{11}{2} - \frac{\pi}{4} + \left(1 - \frac{\pi}{4}\right) = \frac{13}{2} - \frac{\pi}{2}$$

Therefore, the absolute minimum value is $g(-4) = -1$.

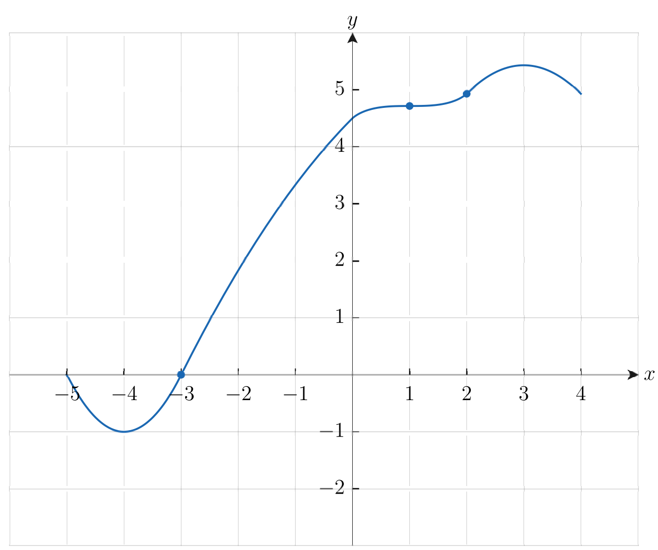
(d) Find all the values of x in the open interval $(-5, 4)$ at which the graph of g has a point of inflection.



Graph of f

$g'(x) = f(x)$

g has a point of inflection at $x = -3$, $x = 1$, and at $x = 2$ because the graph of $g'(x) = f(x)$ changes from increasing to decreasing or decreasing to increasing at these values.



Graph of g