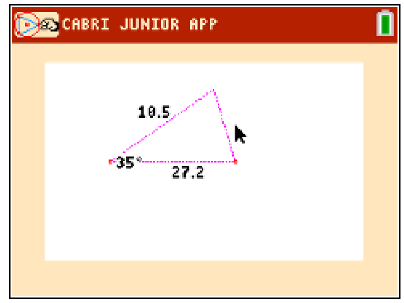
IB Aligned

Laws of Sines and Cosines



Math Objectives

- Students will derive, discuss and apply the Law of Sines and the Law of Cosines.
- Students will practice applying these laws, and the area of a triangle using trigonometry, to real world situations.
- Students will try to make a connection with how to understand these topics in IB Mathematics courses and on their final assessments.

Vocabulary

- Proofs
- Oblique Triangles
- Law of Sines
- Law of Cosines
- Right Triangle Trigonometry

About the Lesson

- This lesson is aligning with the curriculum of IB Mathematics Applications and Interpretations SL/HL and IB Mathematics Approaches and Analysis SL/HL
- This falls under the IB Mathematics Content Topic 3 Geometry and Trigonometry:

  3.2 of the Core Curriculum:

  (a) Use of sin, cos, and tan ratios to find sides and angles of right angled triangles

  (b) The sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

  (c) The cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A \quad or$

  $$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}$$

  (d) Area of a triangle in the form $\frac{1}{2}\,ab\sin C$

  3.3 of the Core Curriculum:

  (a) Applications of right and non-right-angled trig including Pythagorean theorem

  As a result, students will:

  - Apply this information to real world situations.

Teacher Preparation and Notes.

- This activity is done with the use of the TI-84 family as an aid to the problems.

Tech Tips:

- This activity includes screen captures taken from the TI-84 Plus CE. It is also appropriate for use with the rest of the TI-84 Plus family. Slight variations to these directions may be required if using other calculator models.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Files:

Student Activity

Laws of Sines and Cosines_Student-84CE.pdf

Laws of Sines and Cosines_Student-84CE.doc

Activity Materials

- Compatible TI Technologies: TI-84 Plus*, TI-84 Plus Silver Edition*, TI-84 Plus C Silver Edition, TI-84 Plus CE

** with the latest operating system (2.55MP) featuring MathPrint™ functionality.*

Throughout history, mathematicians from Euclid to al-Kashi to Viète have derived various formulas to calculate the sides and angles of non-right (oblique) triangles. al-Kashi used these methods to find the angles between the stars back in the 15th century. Both the famous Laws of Sines and Cosines are used extensively in surveying, navigation, and other situations that require triangulation of non-right triangles. In this activity, you will explore the proofs of the Laws, investigate various cases where they are used, and apply them to solve problems.

Teacher Tip: If time permits, the teacher can demonstrate how to create and manipulate sides and angles of triangles using Cabri Jr. on the handheld. This may help students in understanding the discussion questions about moving angle C.

Problem 1 – Review of Geometry

(a) Discuss with a classmate what SAS, ASA, SAA, SAS, SSS, and SSA mean. Share your results with the class.

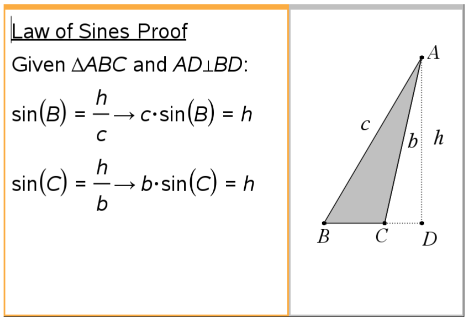
Possible Discussion: These abbreviations represent the given information in a triangle as part of the 3 sides and 3 angles. For example, SAS is side-angle-side which means you are given two sides of a triangle and the included angle between those two sides.

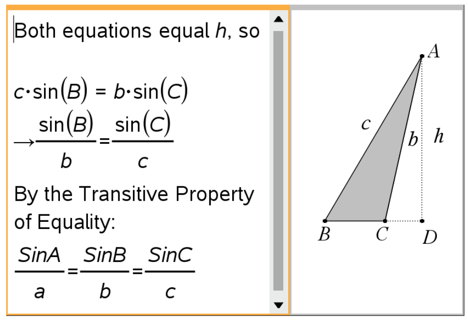
(b) Explain why one of these abbreviations does not always work.

Possible Explanation: SSA cannot be used to prove two triangles congruent, unless it is used in conjunction with right triangles in the hypotenuse-leg scenario. It is also referred to as the ambiguous case for the Law of Sines that results in possible 0, 1, or 2 solution triangles due to the lack of information given.

To find the side lengths and angles of various oblique triangles, we need three pieces of information. There are four cases of triangles that you will investigate:
Case 1: ASA (Law of Sines)
Case 2: SAA (Law of Sines)
Case 3: SAS (Law of Cosines)
Case 4: SSS (Law of Cosines)

Problem 2 – Proof of the Law of Sines





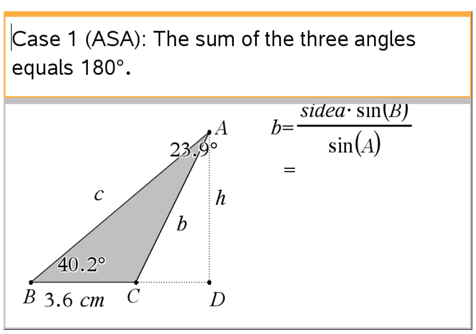
The angle C refers to the angle ACD.

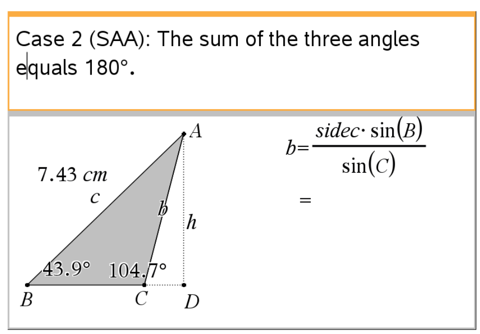
(a) Imagine you could move point C so that it is an acute angle, discuss if the Law of Sines still holds.

Possible Discussion: Yes, the Law of Sines still holds true as the sine of the angles are still in proportion with their opposite sides.

Teacher Tip: Teachers may want to extend this discussion with the students to see if they can find any scenario when it would not work out. Get your students talking!

Problem 3 – ASA and SAA cases





(a) Case 1: b = ________

Solution: 5.72

Case 2: b = ________

Solution: 5.32

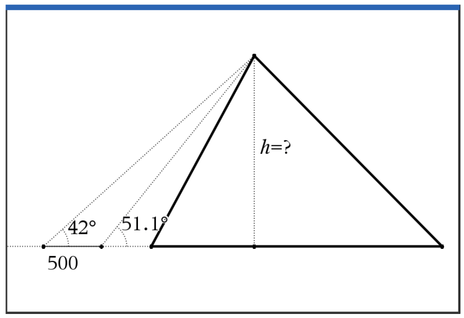
(b) Discuss if moving point C and changing its angle affects your answer to the length of *b.*

Possible Explanation: Yes, the Law of Sines still holds true as the sine of the angles are still in proportion with their opposite sides.

Problem 4 – Law of Sines Problem

Use the Law of Sines to solve the following problem:

A surveyor took two angle measurements to the peak of the mountain 500m apart. Find the height of the mountain.



Solution: Step 1: Find the supplement to $51.1°$ $180° - 51.1° = 128.9°$

Step 2: Using this angle, find the third angle of the triangle on the left.

$$180° - 128.9° - 42° = 9.1°$$

Step 3: You now have a ASA scenario to use the Law of Sines with to find the hypotenuse of the right triangle on the right.

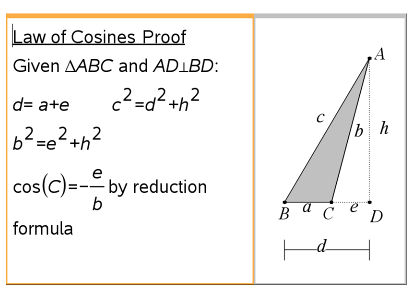
$$\frac{500}{\sin 9.1°} = \frac{x}{\sin 42°} \quad \rightarrow \quad x = \frac{500 \sin 42°}{\sin 9.1°} \quad \rightarrow \quad x = 2115.39$$

Step 4: Using right triangle trig, find the height of the mountain.

$$\sin 51.1° = \frac{h}{2115.39} \quad \rightarrow \quad h = 2115.39 \sin 51.1° \quad \rightarrow \quad h = 1646.28\ m$$

Problem 5 – Proof of the Law of Cosines

Use the 4 pieces of information below and algebra to complete the proof.



A. Substitute 1 into 2 and simplify.
B. Solve 3 for h^2 and 4 for e.
C. Substitute the results from B into A.

1. $d = a + e$
2. $c^2 = d^2 + h^2$
3. $b^2 = e^2 + h^2$
4. $\cos(C) = -\frac{e}{b}$

The result is the Law of Cosines.

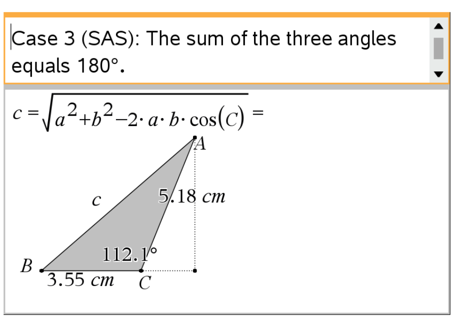
Solution/Work: A. $c^2 = (a+e)^2 + h^2 \quad \rightarrow \quad c^2 = a^2 + 2ae + e^2 + h^2$
B. $h^2 = b^2 - e^2 \quad and \quad e = -b\cos C$
C. $c^2 = a^2 + 2a(-b\cos C) + e^2 + (b^2 - e^2)$
$c^2 = a^2 + b^2 - 2ab\cos C$

(a) Imagine you could move point *C* so that it is an acute angle, discuss with a classmate if the Law of Cosines still holds true.

Possible Discussion: Yes the Law of Cosines still holds true as the relationship between the sides and angles remains the same.

Teacher Tip: This is a good place to have students do the work at the board and have them explain their procedures and results to the class.

Problem 6 – SAS and SSS Cases



(a) Case 3: *c* = _______

Solution: 7.3

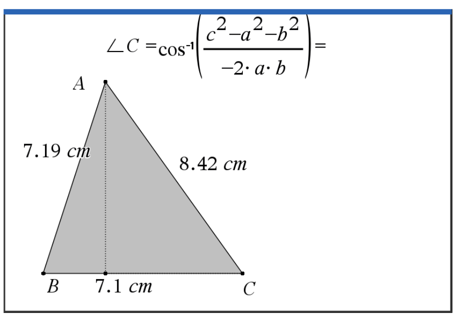
(b) Imagine you could move point *C*. Discuss with a classmate how moving point *C* may affect your answer.

Possible Discussion: The *c* value changes based on the size of angle C but the Law of Cosines still holds true.

Case 4 (SSS): In this triangle, all of the lengths of the sides are known, but none of the angles measures are known. To calculate the measure of an angle, the Law of Cosines must be rearranged.

(c) Name the trig function that must be used in Case 4 to calculate the angle.

Solution: Inverse cosine (cos^{-1})



(d) Case 4: m∠*C* = ______

Solution: 54.4°

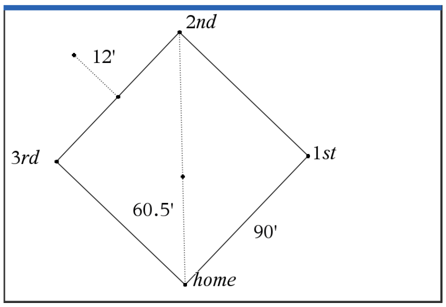
(e) Imagine you could move point *C*. Discuss with a classmate how moving point *C* may affect your answer.

Possible Discussion: As sides *a* and *b* change, so does angle *C*, but the Law of Cosines still holds true.

Problem 7 – Law of Cosines Problem

Use the Law of Cosines and the diagram below to solve the following problem.

A Major League baseball diamond is a square with each side measuring 90 feet. The pitching mound is located 60.5 feet from home plate on a line joining home plate and second base.



a) Find how far the pitching mound is to first base. Also find how far the mound is to Second base.

Solution: Using the fact that the line connecting home plate and second base bisects the angle at home plate, we can use the $45°$ to give a SAS scenario and use the Law of Cosines.

$$SAS: \quad Mound\ to\ 1st\ base\ distance = \sqrt{90^2 + 60^2 - 2(90)(60)\cos 45} = 63.7\ ft.$$

To find the distance from the mound to second base, we can use the distances from home to first and first to second (90 ft.) and find the distance from home to second using the Pythagorean theorem, as the angle at first base in $90°$. Then, subtract that distance with the distance from the mound to home plate (60.5 ft.)

$$home\ to\ second\ distance = \sqrt{90^2 + 90^2} - 60.5 = 66.8\ ft.$$

b) Facing home plate, find the angle the pitcher will need to turn to face first base.

Solution: This is a SSS scenario (Law of Cosines) since we have the distance from home to the mound (60.5 ft.), from home to first base (90 ft), and we found the distance from the mound to first base (63.7 ft.).

$$\cos C = \frac{63.7^2 + 60.5^2 - 90^2}{2(63.7)(60.5)} \quad \rightarrow \quad C = 92.8°$$

c) If a short stop is standing in the middle of 2nd and 3rd base and 12ft into the outfield, find how far the player is standing from home plate where the ball is to be thrown.

Solution: This is a multi-step problem where you create two triangles with the given information. The first triangle is created by connecting second base with the midpoint between second and third base and with home plate. It is a SAS scenario (Law of Cosines) and you would find the side connecting home plate with the midpoint of second and third base. Once that is found you would create a triangle with that distance, the distance between the midpoint of second and third and the 12 ft. distance into the outfield and the line connecting that outfield point to home plate. You have two sides of the triangle, but are missing the angle between them. You will use the first triangle you created with a SSS scenario (Law of Cosines) to find the angle at the midpoint. Find this angle's supplement and add 90° to it to find the missing angle we need in the second triangle. Now we have another SAS scenario and will use the Law of Cosines again to find the distance from the short stop to home plate.

$$home\ to\ midpoint\ distance = \sqrt{45^2 + (90\sqrt{2})^2 - 2(45)(90\sqrt{2})\cos 45°} = 100.623\ ft.$$

$$Angle\ at\ the\ midpoint\ of\ first\ triangle:\ \cos\theta = \frac{45^2 + 100.623^2 - 90\sqrt{2}^2}{2(45)(100.623)} \qquad \theta = 116.565°$$

$$180° - 116.565° = 63.4° + 90° = 153.435°$$

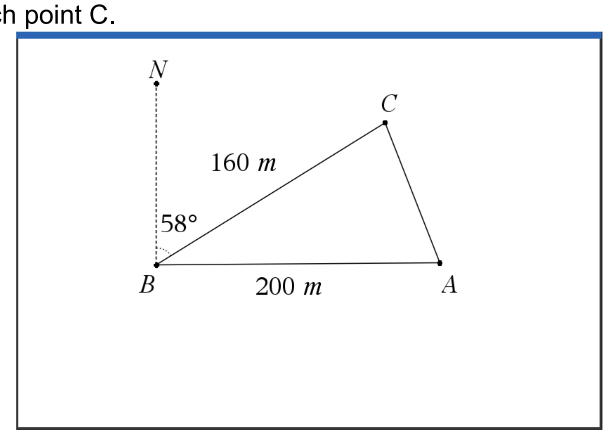
$$short\ stop\ to\ home\ distance = \sqrt{12^2 + 100.623^2 - 2(12)(100.623)\cos 153.435°}$$

$$= 111.485\ ft.$$

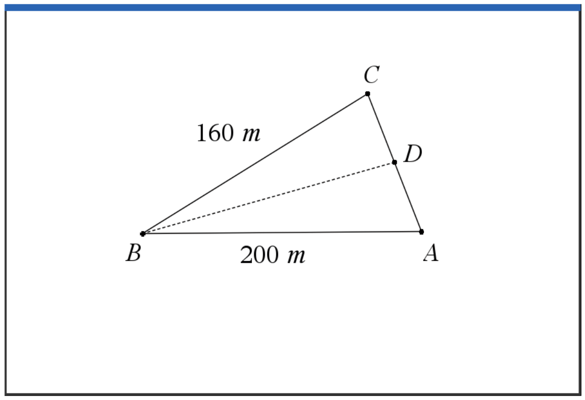
Further IB Applications

Dwight is reimagining his beet farm. He wants to place posts A, B, and C according to his diagram below. These posts will mark off a triangular piece of his land optimal for growing the finest beets in the world.

From point A, he walks due west 200 meters to point B. From point B, he walks 160 meters on a bearing of 058° to reach point C.



Dwight wants to divide the land into two sections to change his planting patterns and test which produce better beets. He will put a post at point D, which will be between A and C. He wants the boundary BD to divide the land so he will have two equal areas. See the diagram below.



(a) Find the distance from A to C.

Solution: $90° - 58° = 32°$

SAS (Law of Cosines): $\sqrt{160^2 + 200^2 - 2(160)(220)\cos 32°} = 106.4186\ldots = 106\ m$

(b) Find the area of the entire triangular ABC piece of land.

Solution: $Area = \frac{1}{2}(160)(200)\sin 32° = 8478.7082\ldots = 8480\ m^2$

(c) Find the measure of angle A.

Solution: $SSS \quad \cos A = \frac{200^2 + 106.4186^2 - 160^2}{2(200)(106.4186)} \quad A = 52.8°$

(d) Find the distance from point B to point D.

Solution: $DA = \frac{1}{2}CA = \frac{1}{2}(106.4186) = 53.2093$

$(SAS)\ BD = \sqrt{200^2 + 53.2093^2 - 2(200)(53.2093)\cos 52.8°} = 173.09855\ldots = 173\ m$

Teacher Tip: Please know that in this activity there is a lot of time dedicated to students talking with one another and sharing their thoughts with the class. The goal here is to not only review the Laws of Sines and Cosines, but also to generate discussion.

***Note: This activity has been developed independently by Texas Instruments and aligned with the IB Mathematics curriculum, but is not endorsed by IB™. IB is a registered trademark owned by the International Baccalaureate Organization.*