Monday Night Calculus, October 4, 2021

1. Show algebraically, using the alternate form of the derivative, why $g(x)$ is not differentiable at $x = 1$. (April Corbin)

$$g(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ x & \text{if } x > 1 \end{cases}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

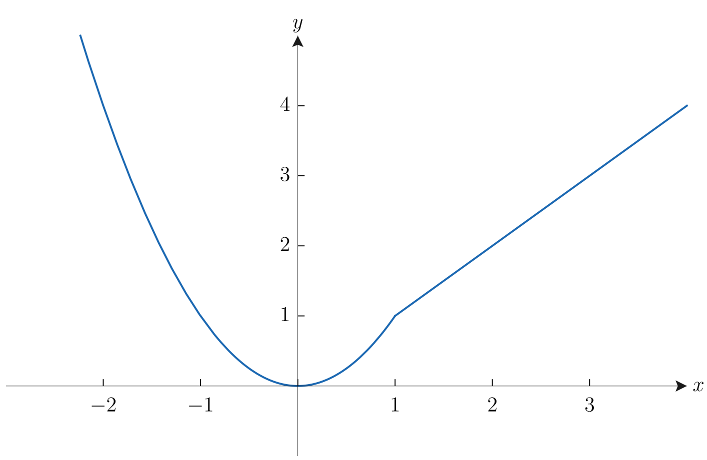
$$g'(1) = \lim_{x \to 1} \frac{g(x) - g(1)}{x - 1}$$

$$\lim_{x \to 1^+} \frac{g(x) - g(1)}{x - 1} = \lim_{x \to 1^+} \frac{x - 1^2}{x - 1} = \lim_{x \to 1^+} 1 = 1$$

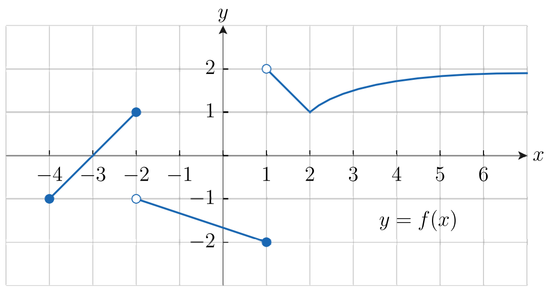
$$\lim_{x \to 1^-} \frac{g(x) - g(1)}{x - 1} = \lim_{x \to 1^-} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1^-} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1^-} (x + 1) = 2$$

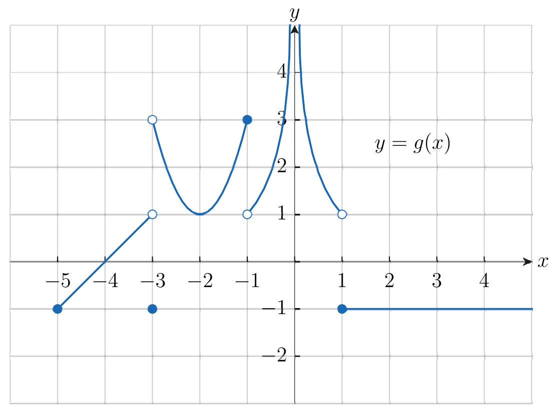
Therefore, $\lim_{x \to 1} \frac{g(x) - g(1)}{x - 1}$ does not exist.

The function g is not differentiable at $x = 1$.



2. The graphs of f and g are shown below. (Yasemin Gunes)





Find $\lim\limits_{x \to -3} [f(x) \cdot g(x)]$ or show that it does not exist.

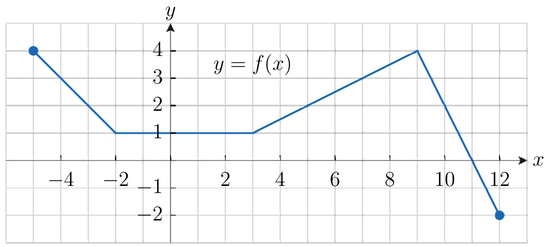
$$\lim_{x \to -3} [f(x) \cdot g(x)] \stackrel{?}{=} \lim_{x \to -3} f(x) \cdot \lim_{x \to -3} g(x)$$

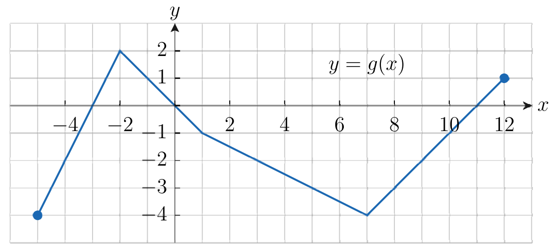
$$\lim_{x \to -3^-} [f(x) \cdot g(x)] = \lim_{x \to -3^-} f(x) \cdot \lim_{x \to -3^-} g(x) =$$

$$\lim_{x \to -3^+} [f(x) \cdot g(x)] = \lim_{x \to -3^+} f(x) \cdot \lim_{x \to -3^+} g(x) =$$

Therefore, $\lim\limits_{x \to -3} [f(x) \cdot g(x)] =$

3. The graphs of f and g are shown below. (Katie Rich via Bryan Passwater)





The function A is defined by $A(x) = g(f(x))$. Find $A'(1)$.

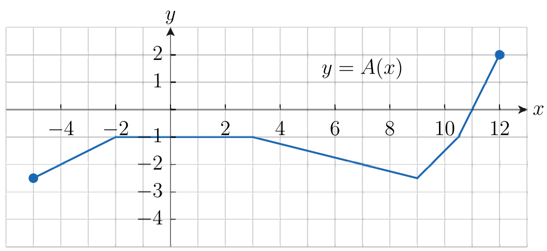
$$A'(x) = g'(f(x)) \cdot f'(x) \quad \Rightarrow \quad A'(1) \stackrel{?}{=} g'(f(1)) \cdot f'(1) = g'(1) \cdot 0 =$$

The Chain Rule

If f is differentiable at c and g is differentiable at $f(c)$, then the composite function $A = g \circ f$ defined by $A(x) = g(f(x))$ is differentiable at c and $A'(c)$ is given by the product $A'(c) = g'(f(c)) \cdot f'(c)$.

f differentiable at 1?

g differentiable at $f(1)$?



Therefore, $A'(1) =$

4. Use implicit differentiation to find $\dfrac{dy}{dx}$ at the given point. (Bob Destefano)

$x^2 = (4x^3y^3 + 3)^2$ at $(1, -1)$

$$2x = 2(4x^3y^3 + 3)(12x^2y^3 + 4x^3 3y^2 y')$$

$$\frac{x}{4x^3y^3 + 3} = 12x^2y^3 + 4x^3 3y^2 y'$$

$$12x^3y^2y' = \frac{x}{4x^3y^3 + 3} - 12x^2y^3$$

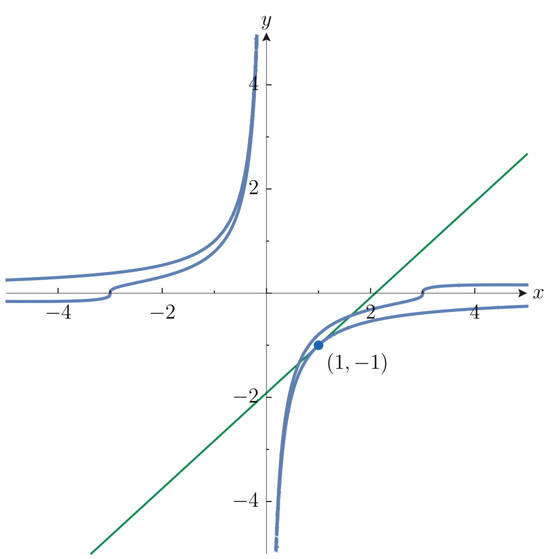
$$y' = \frac{1}{(12x^2y^2)(4x^3y^3 + 3)} - \frac{y}{x}$$

$$y'\Big|_{(x,y)=(1,-1)} = \frac{1}{(12 \cdot 1^2(-1)^2)(4 \cdot 1^3(-1)^3 + 3)} - \frac{-1}{1} = \frac{1}{(12)(-1)} + 1 = -\frac{1}{12} + 1 = \frac{11}{12}$$

$$2x = 2(4x^3y^3 + 3)(12x^2y^3 + 4x^3 3y^2 y') \qquad \text{Let } (x, y) = (1, -1).$$

$$1 = (4 \cdot 1^3(-1)^3) + 3)(12 \cdot 1^2(-1)^3 + 12 \cdot 1^3(-1)^2 y')$$

$$1 = (-1)(-12 + 12y') \;\Rightarrow\; -1 = -12 + 12y' \;\Rightarrow\; 11 = 12y' \;\Rightarrow\; y' = \frac{11}{12}$$



5. If f is a differentiable function such that $f(3) = 8$ and $f'(3) = 5$, which of the following statements could be false? (Rachel Brusda)

(A) $\lim_{x\to 3} f(x) = 8$

(B) $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x)$

(C) $\lim_{x\to 3} \dfrac{f(x) - 8}{x - 3} = 5$

(D) $\lim_{h\to 0} \dfrac{f(3+h) - 8}{h} = 5$

(E) $\lim_{x\to 3} f'(x) = 5$

It is possible to have a function f defined for all real numbers such that f is a differentiable function everywhere on its domain but the derivative f' is not a continuous function.

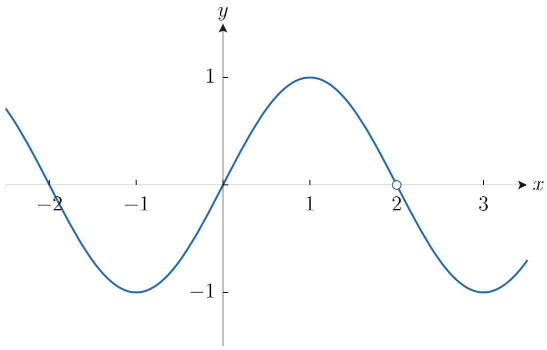
Equivalently: A differentiable function on the real numbers need not be a continuously differentiable function.

Classic example:

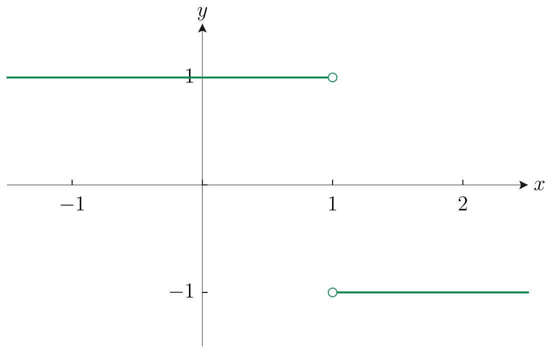
$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

6. The graphs of the functions f and g are shown in the figures. (Michael Cole via Tony Record)



Graph of f



Graph of g

Which of the following statements is false?

(A) $\lim_{x \to 2} f(x) = 0$

(B) $\lim_{x \to 1} g(x)$ does not exist

(C) $\lim_{x \to 2} [f(x) \cdot g(x-1)]$ does not exist

(D) $\lim_{x \to 2} [f(x-1) \cdot g(x)]$ exists