

TI in Focus: AP[®] Calculus

2019 AP[®] Calculus Exam: BC-2

Technology Solutions and Problem Extensions

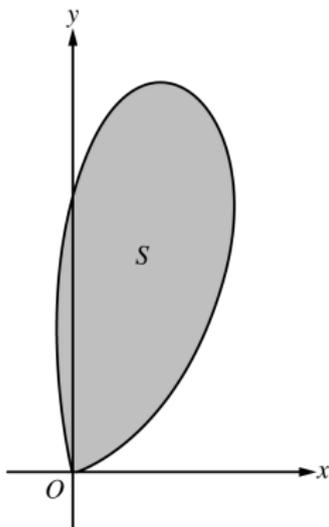
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Outline

- (1) Free Response Question
- (2) Scoring Guidelines
- (3) Solutions in greater detail
- (4) Solutions using technology
- (5) Problem Extensions



2. Let S be the region bounded by the graph of the polar curve $r(\theta) = 3\sqrt{\theta} \sin(\theta^2)$ for $0 \leq \theta \leq \sqrt{\pi}$, as shown in the figure above.

- (a) Find the area of S .
- (b) What is the average distance from the origin to a point on the polar curve $r(\theta) = 3\sqrt{\theta} \sin(\theta^2)$ for $0 \leq \theta \leq \sqrt{\pi}$?
- (c) There is a line through the origin with positive slope m that divides the region S into two regions with equal areas. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of m .
- (d) For $k > 0$, let $A(k)$ be the area of the portion of region S that is also inside the circle $r = k \cos \theta$. Find $\lim_{k \rightarrow \infty} A(k)$.

$$(a) \frac{1}{2} \int_0^{\sqrt{\pi}} (r(\theta))^2 d\theta = 3.534292$$

The area of S is 3.534.

$$(b) \frac{1}{\sqrt{\pi} - 0} \int_0^{\sqrt{\pi}} r(\theta) d\theta = 1.579933$$

The average distance from the origin to a point on the curve $r = r(\theta)$ for $0 \leq \theta \leq \sqrt{\pi}$ is 1.580 (or 1.579).

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

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$$(c) \quad \tan \theta = \frac{y}{x} = m \Rightarrow \theta = \tan^{-1} m$$

$$\frac{1}{2} \int_0^{\tan^{-1} m} (r(\theta))^2 d\theta = \frac{1}{2} \left(\frac{1}{2} \int_0^{\sqrt{\pi}} (r(\theta))^2 d\theta \right)$$

(d) As $k \rightarrow \infty$, the circle $r = k \cos \theta$ grows to enclose all points to the right of the y -axis.

$$\begin{aligned} \lim_{k \rightarrow \infty} A(k) &= \frac{1}{2} \int_0^{\pi/2} (r(\theta))^2 d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} (3\sqrt{\theta} \sin(\theta^2))^2 d\theta = 3.324 \end{aligned}$$

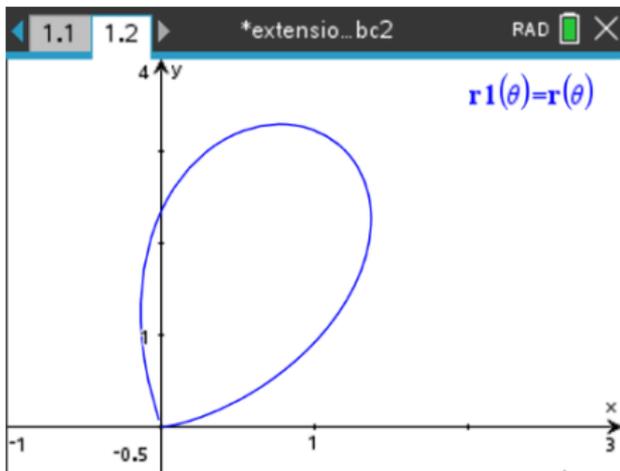
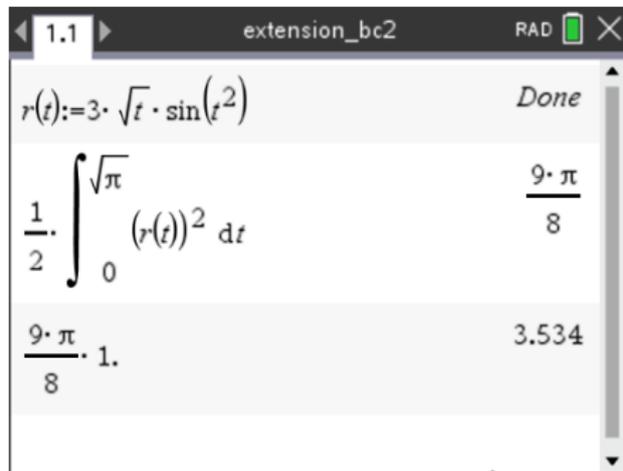
3 : $\left\{ \begin{array}{l} 1 : \text{equates polar areas} \\ 1 : \text{inverse trigonometric function} \\ \text{applied to } m \text{ as limit of} \\ \text{integration} \\ 1 : \text{equation} \end{array} \right.$

2 : $\left\{ \begin{array}{l} 1 : \text{limits of integration} \\ 1 : \text{answer with integral} \end{array} \right.$

Part (a)

Solution

$$\frac{1}{2} \int_0^{\sqrt{\pi}} [r(\theta)]^2 d\theta = 3.534$$



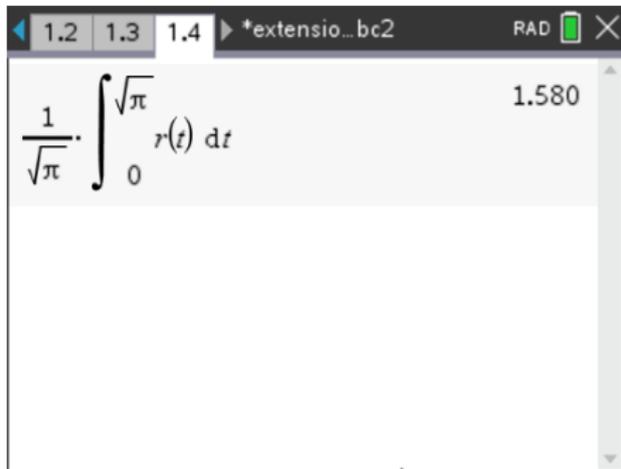
Example 1 The Unabridged Version

Consider the polar curve $r(\theta) = 3\sqrt{\theta} \sin(\theta^2)$.

- If possible, sketch the complete polar curve. If not, explain why not?
- Let R_1 be the region bounded by the polar curve for $\sqrt{\pi} \leq \theta \leq \sqrt{2\pi}$. Find the area of R_1 and sketch a graph to illustrate this result.
- Let R_2 be the region bounded by the polar curve for $\sqrt{4\pi} \leq \theta \leq \sqrt{5\pi}$. Find the area of R_2 and sketch a graph to illustrate this result.
- Use your results from parts (b) and (c), and part (a) from the Free Response Question to state a general result about the area enclosed by loops of this polar curve.

Part (b)**Solution**

$$\frac{1}{\sqrt{\pi} - 0} \int_0^{\sqrt{\pi}} r(\theta) d\theta = 1.580$$



The screenshot shows a TI-84 Plus calculator interface. At the top, the window title is "*extensio...bc2" and the mode is set to "RAD". The display shows the integral expression $\frac{1}{\sqrt{\pi}} \cdot \int_0^{\sqrt{\pi}} r(t) dt$ on the left and the numerical result "1.580" on the right. The calculator interface includes navigation arrows, a mode indicator, and a close button.

Part (b)

Alternate Solution

Distance from any point (x, y) to the origin: $\sqrt{x^2 + y^2}$

Write x and y in terms of r and θ : $x = r(\theta) \cos \theta$ $y = r(\theta) \sin \theta$

Find the average distance from the origin for $0 \leq \theta \leq \sqrt{\pi}$.

$$\frac{1}{\sqrt{\pi}} \int_0^{\sqrt{\pi}} \sqrt{[r(\theta) \cos \theta]^2 + [r(\theta) \sin \theta]^2} d\theta = 1.580$$

The screenshot shows a TI-84 Plus calculator interface. At the top, the window title is "*extensio...bc2" and the mode is set to "RAD". The display shows the integral expression $\frac{1}{\sqrt{\pi}} \int_0^{\sqrt{\pi}} \sqrt{(r(t) \cdot \sin(t))^2 + (r(t) \cdot \cos(t))^2} dt$ and the numerical result "1.580".

Example 2 Arc Length

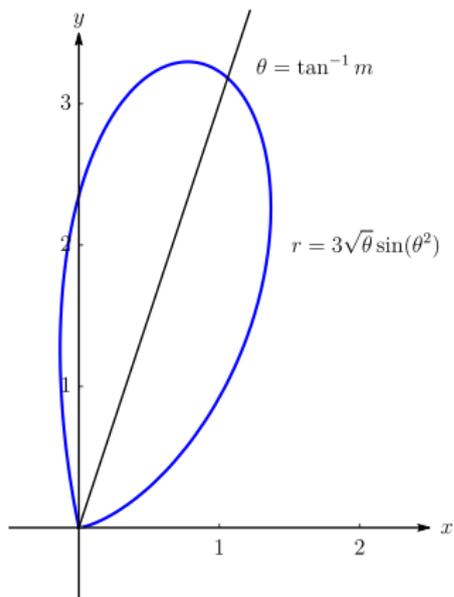
Consider the polar curve $r(\theta) = 3\sqrt{\theta} \sin(\theta^2)$.

- (a) Find the length of the polar curve for $0 \leq \theta \leq \sqrt{\pi}$.
- (b) Find the length of the polar curve for $\sqrt{\pi} \leq \theta \leq \sqrt{2\pi}$.

Part (c)**Solution**

Equation of the line: $y = mx$

$$\tan \theta = \frac{y}{x} = m \Rightarrow \theta = \tan^{-1} m$$



$$\frac{1}{2} \int_0^{\tan^{-1} m} [r(\theta)]^2 d\theta = \frac{1}{2} \left[\frac{1}{2} \int_0^{\sqrt{\pi}} [r(\theta)]^2 d\theta \right] = \frac{3.534292}{2}$$

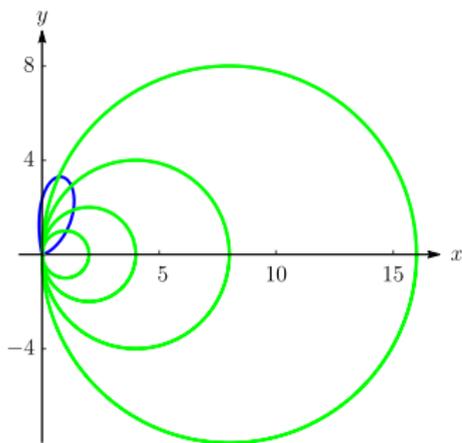
Example 3 The Dividing Line

Use technology to find the value of m as described in part (c).

Part (d)

Solution

For $k > 0$, $A(k)$ is the area of the portion of S inside the circle $r = k \cos \theta$.



| Equation | Value |
|--|-------|
| $\text{nSolve}(3 \cdot \sqrt{t} \cdot \sin(t^2) = 8 \cdot \cos(t), t)$ | 1.164 |
| $\text{nSolve}(3 \cdot \sqrt{t} \cdot \sin(t^2) = 16 \cdot \cos(t), t)$ | 1.359 |
| $\text{nSolve}(3 \cdot \sqrt{t} \cdot \sin(t^2) = 32 \cdot \cos(t), t)$ | 1.477 |
| $\text{nSolve}(3 \cdot \sqrt{t} \cdot \sin(t^2) = 100 \cdot \cos(t), t)$ | 1.545 |

Part (d)

Solution

$$\lim_{k \rightarrow \infty} A(k) = \frac{1}{2} \int_0^{\pi/2} [r(\theta)]^2 d\theta = 3.324$$

The screenshot shows a TI-84 Plus calculator interface. At the top, the window title is '*extensio...bc2' and the mode is 'RAD'. The screen displays the integral expression $\frac{1}{2} \int_0^{\frac{\pi}{2}} (r(t))^2 dt$ on the left. To its right, the expression $\frac{-9 \cdot \left(2 \cdot \sin\left(\frac{\pi^2}{2}\right) - \pi^2 \right)}{32}$ is shown. Below the integral, the expression $\frac{-9 \cdot \left(2 \cdot \sin\left(\frac{\pi^2}{2}\right) - \pi^2 \right)}{32} \cdot 1$ is displayed, with the numerical result '3.324' shown to its right.

Example 4 More Loops

Consider the polar equation $r(\theta) = \sqrt{5 \sin(4\theta)}$.

- Sketch a complete graph of the polar curve r .
- Let S be the region enclosed by the graph of r in the first quadrant. Find the area of S .
- There is a line through the origin with positive slope m that divides the region S into two regions of equal areas. Find the value of m .

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