TI in Focus: AP® Calculus

2019 AP® Calculus Exam: BC-5
Partial Fractions

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Outline

(1) The General Method
(2) Long Division
(3) Three Distinct Linear Factors
(4) The Difference of Two Squares
(5) Examples

Background

- Objective: Develop a method for integrating a rational function (quotient of two polynomials).
- The method of writing a rational function as a sum of simpler fractions: partial fractions.
- Example:

$$\frac{2}{x-3}+\frac{1}{x+2}=\frac{2(x+2)+(x-3)}{(x-3)(x+2)}=\frac{3x+1}{x^2-x-6}$$

$$\int \frac{3x+1}{x^2-x-6}\,dx=\int\left[\frac{2}{x-3}+\frac{1}{x+2}\right]dx$$

$$=2\ln|x-3|+\ln|x+2|+C$$

The Method of Partial Fractions

- Consider a rational function: $f(x) = \dfrac{P(x)}{Q(x)}$

  where P and Q are polynomials.

- Recall: if $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $a_n \neq 0$, then the degree of P is n and we write $\deg(P) = n$

- If $\deg(P) < \deg(Q)$ then f is *proper*.

  And it is possible to write f as a sum of simpler fractions, that is, rational functions with denominators of lower degree.

- If $\deg(P) \geq \deg(Q)$, then f is *improper*.

  Use long division until a remainder $R(x)$ is obtained such that $\deg(R) < \deg(Q)$

  $$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

Example 1 Long Division

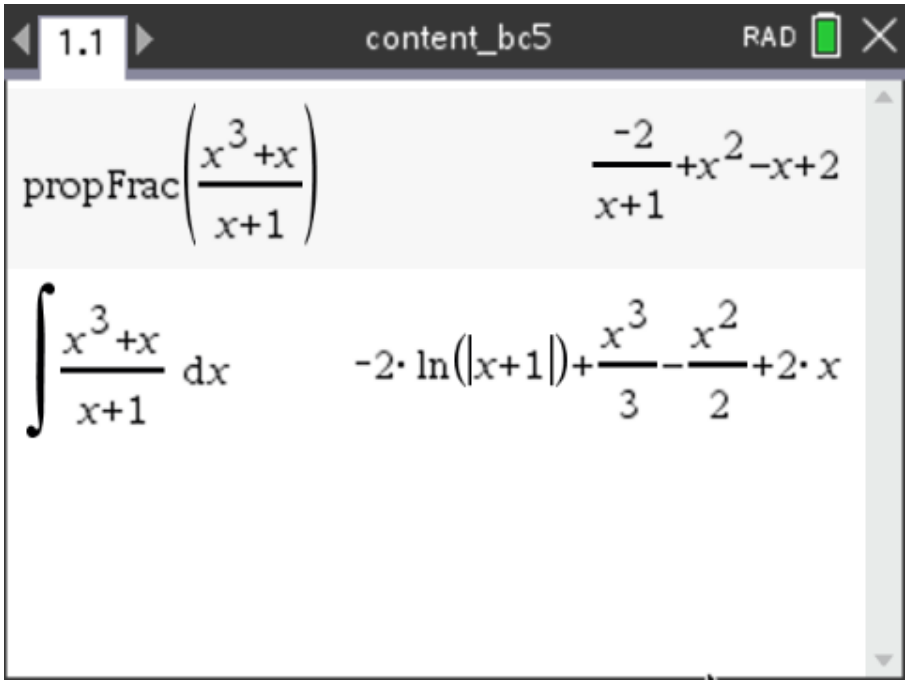
Find $\displaystyle\int \frac{x^3+x}{x+1}\,dx$

Solution

$$\int \frac{x^3+x}{x+1}\,dx = \int \left[x^2 - x + 2 - \frac{2}{x+1}\right]\,dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + 2x - 2\ln|x+1| + C$$

Technology Solution



Factor $Q(x)$ Completely

- Any polynomial Q can be factored into products of expressions of the form

$$(ax+b)^m \quad \text{and} \quad (ax^2+bx+c)^n$$

  where ax^2+bx+c is irreducible.

- Example: $x^4-81=(x^2-9)(x^2+9)=(x-3)(x+3)(x^2+9)$

- For *each* (linear) factor of the form $(ax+b)^m$, the partial fraction decomposition must include the following m fractions.

$$\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_m}{(ax+b)^m}$$

- For *each* (quadratic) factor of the form $(ax^2+bx+c)^n$, the partial fraction decomposition must include the following n fractions.

$$\frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

- A theorem in algebra guarantees that for any proper rational function we can always find this partial fraction decomposition.

Example 2 Three Distinct Linear Factors

Evaluate $\displaystyle\int \frac{x^2+3x-2}{x^3+3x^2-4x}\,dx$

Solution

The degree of the numerator is less than the degree in the denominator: no long division.

Factor the denominator completely:

$$x^3+3x^2-4x = x(x^2+3x-4) = x(x-1)(x+4)$$

Three distinct linear factors. The PFD has the form:

$$\frac{x^2+3x-2}{x^3+3x^2-4x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+4}$$

Solution (Continued)

Find the least common denominator on the right side:

$$\frac{x^2+3x-2}{x^3+3x^2-4x} = \frac{A(x-1)(x+4)+Bx(x+4)+Cx(x-1)}{x^3+3x^2-4x}$$

Two rational functions that are equal, and their denominators are the same. Their numerators must be the same. Equate the numerators.

$$\begin{aligned} x^2+3x-2 &= A(x-1)(x+4)+Bx(x+4)+Cx(x-1) \\ &= (A+B+C)x^2+(6A+4B-2C)x+8A \end{aligned}$$

Two polynomials that are equal, so their coefficients must be the same.

Equate the coefficients.

Solution (Continued)

System of equations: three equations in three unknowns.

$$\begin{cases} A + \ \ B + C = 1 \\ 3A + 4B - C = 3 \\ -4A \qquad\qquad = -2 \end{cases}$$

System Solution: $A = \dfrac{1}{2}, \quad B = \dfrac{2}{5}, \quad C = \dfrac{1}{10}$

$$\int \frac{x^2 + 3x - 2}{x^3 + 3x^2 - 4x}\, dx = \int \left[\frac{1}{2}\,\frac{1}{x} + \frac{2}{5}\,\frac{1}{x-1} + \frac{1}{10}\,\frac{1}{x+4}\right] dx$$

$$= \frac{1}{2}\ln|x| + \frac{2}{5}\ln|x-1| + \frac{1}{10}\ln|x+4| + C$$

Technology Solution

1.1 | 2.1 | 2.2 | *content_bc5 | RAD

$$f(x) := \frac{x^2+3\cdot x-2}{x^3+3\cdot x^2-4\cdot x}$$

Done

$$\text{expand}(f(x)) \qquad \frac{1}{10\cdot(x+4)}+\frac{2}{5\cdot(x-1)}+\frac{1}{2\cdot x}$$

1.1 | 2.1 | 2.2 | *content_bc5 | RAD

$$\int f(x)\,dx \qquad \frac{\ln\left((x-1)^4\cdot|x|^5\cdot|x+4|\right)}{10}$$

Note: Alternative method for finding the constants.

This equation is true for every value of x:

$$x^2 + 3x - 2 = A(x-1)(x+4) + Bx(x+4) + Cx(x-1)$$

Select values of x that simplify this equation.
Consider the zero for each linear factor.

$$x = 0: \ -2 = A(-1)(4) \quad \Rightarrow \quad A = \frac{1}{2}$$

$$x = 1: \ 2 = B(1)(5) \quad \Rightarrow \quad B = \frac{2}{5}$$

$$x = -4: \ 2 = C(-4)(-5) \quad \Rightarrow \quad C = \frac{1}{10}$$

This method works nicely if we have distinct linear factors in the PFD.
In other cases: use a combination of techniques to find the constants.
The method of solving a system of equations always works.

Example 3 The Difference of Two Squares

Find $\displaystyle\int \frac{1}{x^2 - a^2}\,dx$ where $a \neq 0$

Solution

Factor the denominator: $x^2 - a^2 = (x-a)(x+a)$

$$\frac{1}{x^2 - a^2} = \frac{A}{x-a} + \frac{B}{x+a}$$

$$1 = A(x+a) + B(x-a)$$

Use the zero of each linear factor for x.

$$x = a: \; 1 = A(2a) \quad \Rightarrow \quad A = \frac{1}{2a}$$

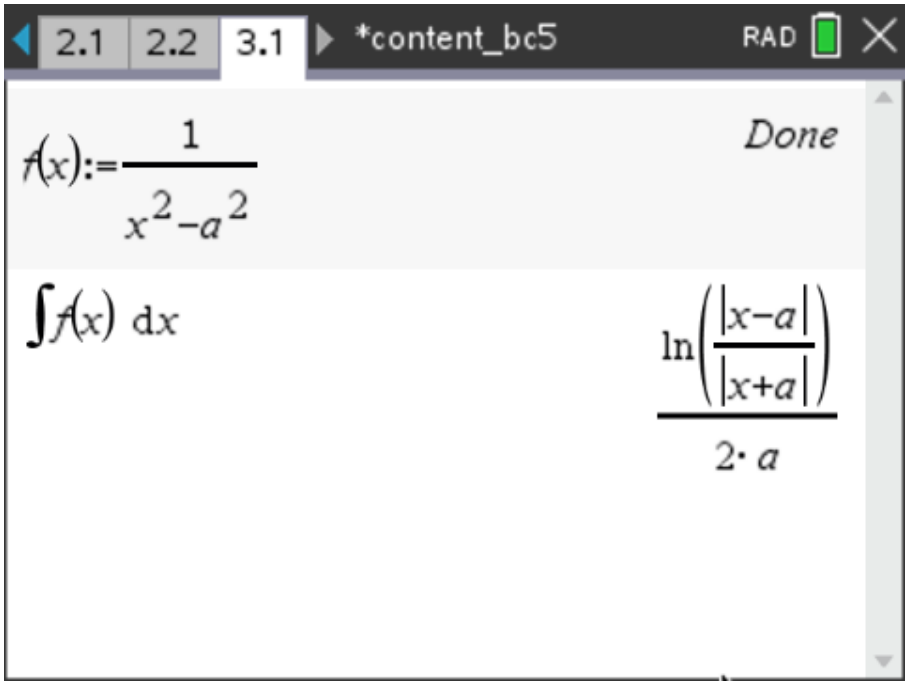
$$x = -a: \; 1 = B(-2a) \quad \Rightarrow \quad B = -\frac{1}{2a}$$

Solution (Continued)

$$\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\int \left[\frac{1}{x-a} - \frac{1}{x+a}\right] dx$$

$$= \frac{1}{2a}\left[\ln|x-a| - \ln|x+a|\right] + C$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$



Example 4 Integrals Involving PFDs

Evaluate each integral.

(a) $\displaystyle\int \frac{7x+1}{3x^2-11x-4}\,dx$

(b) $\displaystyle\int \frac{3x}{10x^2+11x+3}\,dx$

(c) $\displaystyle\int \frac{11}{x^2-4}\,dx$

(d) $\displaystyle\int \frac{2x^2-3x-7}{x^3+6x^2+11x+6}\,dx$

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