

## TI in Focus: AP<sup>®</sup> Calculus

2019 AP<sup>®</sup> Calculus Exam: AB-6

Technology Solutions and Problem Extensions

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## Outline

- (1) Free Response Question
- (2) Scoring Guidelines
- (3) Solutions in greater detail
- (4) Solutions using technology
- (5) Problem Extensions

6. Functions  $f$ ,  $g$ , and  $h$  are twice-differentiable functions with  $g(2) = h(2) = 4$ . The line  $y = 4 + \frac{2}{3}(x - 2)$  is tangent to both the graph of  $g$  at  $x = 2$  and the graph of  $h$  at  $x = 2$ .

(a) Find  $h'(2)$ .

(b) Let  $a$  be the function given by  $a(x) = 3x^3h(x)$ . Write an expression for  $a'(x)$ . Find  $a'(2)$ .

(c) The function  $h$  satisfies  $h(x) = \frac{x^2 - 4}{1 - (f(x))^3}$  for  $x \neq 2$ . It is known that  $\lim_{x \rightarrow 2} h(x)$  can be evaluated using

L'Hospital's Rule. Use  $\lim_{x \rightarrow 2} h(x)$  to find  $f(2)$  and  $f'(2)$ . Show the work that leads to your answers.

(d) It is known that  $g(x) \leq h(x)$  for  $1 < x < 3$ . Let  $k$  be a function satisfying  $g(x) \leq k(x) \leq h(x)$  for  $1 < x < 3$ . Is  $k$  continuous at  $x = 2$ ? Justify your answer.

$$(a) \quad h'(2) = \frac{2}{3}$$

$$(b) \quad a'(x) = 9x^2h(x) + 3x^3h'(x)$$

$$a'(2) = 9 \cdot 2^2 h(2) + 3 \cdot 2^3 h'(2) = 36 \cdot 4 + 24 \cdot \frac{2}{3} = 160$$

1 : answer

3 :  $\begin{cases} 1 : \text{form of product rule} \\ 1 : a'(x) \\ 1 : a'(2) \end{cases}$

(c) Because  $h$  is differentiable,  $h$  is continuous, so  $\lim_{x \rightarrow 2} h(x) = h(2) = 4$ .

$$\text{Also, } \lim_{x \rightarrow 2} h(x) = \lim_{x \rightarrow 2} \frac{x^2 - 4}{1 - (f(x))^3}, \text{ so } \lim_{x \rightarrow 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4.$$

Because  $\lim_{x \rightarrow 2} (x^2 - 4) = 0$ , we must also have  $\lim_{x \rightarrow 2} (1 - (f(x))^3) = 0$ .

$$\text{Thus, } \lim_{x \rightarrow 2} f(x) = 1.$$

Because  $f$  is differentiable,  $f$  is continuous, so  $f(2) = \lim_{x \rightarrow 2} f(x) = 1$ .

Also, because  $f$  is twice differentiable,  $f'$  is continuous, so

$$\lim_{x \rightarrow 2} f'(x) = f'(2) \text{ exists.}$$

Using L'Hospital's Rule,

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{1 - (f(x))^3} = \lim_{x \rightarrow 2} \frac{2x}{-3(f(x))^2 f'(x)} = \frac{4}{-3(1)^2 \cdot f'(2)} = 4.$$

$$\text{Thus, } f'(2) = -\frac{1}{3}.$$

$$4 : \begin{cases} 1 : \lim_{x \rightarrow 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4 \\ 1 : f(2) \\ 1 : \text{L'Hospital's Rule} \\ 1 : f'(2) \end{cases}$$

(d) Because  $g$  and  $h$  are differentiable,  $g$  and  $h$  are continuous, so

$$\lim_{x \rightarrow 2} g(x) = g(2) = 4 \quad \text{and} \quad \lim_{x \rightarrow 2} h(x) = h(2) = 4.$$

Because  $g(x) \leq k(x) \leq h(x)$  for  $1 < x < 3$ , it follows from the squeeze theorem that  $\lim_{x \rightarrow 2} k(x) = 4$ .

Also,  $4 = g(2) \leq k(2) \leq h(2) = 4$ , so  $k(2) = 4$ .

Thus,  $k$  is continuous at  $x = 2$ .

1 : continuous with justification

## Part (a)

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### Solution

- The line tangent to the graph of  $h$  at  $x = 2$  has slope  $h'(2)$ .
- The line  $y = 4 + \frac{2}{3}(x - 2) = \frac{2}{3}x + \frac{8}{3}$  is tangent to the graph of  $h$  at  $x = 2$ .
- Therefore,  $h'(2) = \frac{2}{3}$

## Part (b)

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### Solution

$$a(x) = 3x^3h(x)$$

Definition of  $a$

$$a'(x) = 3 \cdot 3x^2 \cdot h(x) + 3x^3 \cdot h'(x)$$

Product Rule

$$= 9x^2h(x) + 3x^3h'(x)$$

Simplification

$$a'(2) = 9 \cdot 2^2 \cdot h(2) + 3 \cdot 2^3 \cdot h'(2)$$

Find  $a'(2)$

$$= 36 \cdot 4 + 24 \cdot \frac{2}{3} = 160$$

Use values for  $h(2)$  and  $h'(2)$ ; simplify

## Technology Solution

1.1 \*extensio...ab6 RAD

$a(x) := 3 \cdot x^3 \cdot h(x)$  Done

$\frac{d}{dx}(a(x)) \quad 3 \cdot \frac{d}{dx}(h(x)) \cdot x^3 + 9 \cdot h(x) \cdot x^2$

$\frac{d}{dx}(a(x))|_{x=2} \quad 24 \cdot \frac{d}{dx}(h(x)) + 36 \cdot h(2)$

## Part (c)

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### Solution

•  $h$  is differentiable  $\Rightarrow h$  is continuous:  $\lim_{x \rightarrow 2} h(x) = h(2) = 4$

• Because  $\lim_{x \rightarrow 2} h(x)$  can be evaluated using L'Hospital's Rule, and

$$\lim_{x \rightarrow 2} (x^2 - 4) = 0, \text{ then}$$

$$\lim_{x \rightarrow 2} (1 - [f(x)]^3) = 0 = 1 - [f(2)]^3 \Rightarrow f(2) = 1$$

• Use L'Hospital's Rule

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{1 - [f(x)]^3} = \lim_{x \rightarrow 2} \frac{2x}{-3 \cdot [f(x)]^2 \cdot f'(x)} = \lim_{x \rightarrow 2} \frac{4}{-3 \cdot 1^2 \cdot f'(2)} = 4$$

• Solve for  $f'(2)$

$$\frac{4}{-3f'(2)} = 4 \Rightarrow f'(2) = -\frac{1}{3}$$

## Part (d)

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### Solution

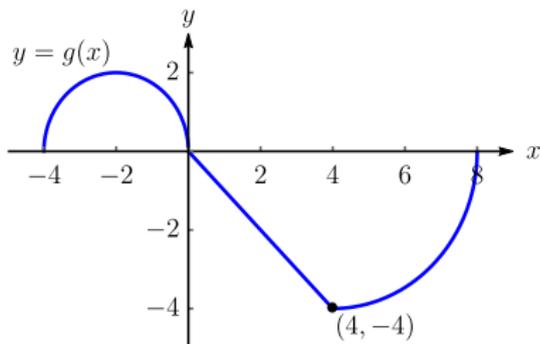
- $g$  and  $h$  are differentiable  $\Rightarrow g$  and  $h$  are continuous.

$$\lim_{x \rightarrow 2} g(x) = g(2) = 4 \quad \text{and} \quad \lim_{x \rightarrow 2} h(x) = h(2) = 4$$

- Since  $g(x) \leq k(x) \leq h(x)$  for  $1 < x < 3$   $\Rightarrow \lim_{x \rightarrow 2} k(x) = 4$
- And  $4 = g(2) \leq k(2) \leq h(2) = 4 \Rightarrow k(2) = 4$
- So,  $\lim_{x \rightarrow 2} k(x) = 4 = k(2) \Rightarrow k$  is continuous at  $x = 2$ .

## Example 1 More with L'Hospital

Let  $h$  be the function given by  $h(x) = e^{-(x-2)^2}$ . The function  $g$  is defined for  $x$  in the interval  $[-4, 8]$ . The graph of  $g$  consists of a half circle, a line segment, and a quarter circle, as shown in the figure.



## Example (Continued)

- (a) Find  $h'(3)$ .
- (b) Find an equation of the tangent line to the graph of  $h$  at the point where  $x = 3$ . Carefully sketch a graph of  $h$  and the tangent line on the same coordinate axes.
- (c) Let  $a$  be the function given by  $a(x) = 4x^4h(x)$ . Write an expression for  $a'(x)$  and find  $a'(2)$ .
- (d) Find  $\lim_{x \rightarrow 2} \frac{x^2h(x) + 2g(x)}{2 \cos(\pi x) - x}$

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