

## TI in Focus: AP<sup>®</sup> Calculus

2020 Mock AP<sup>®</sup> Calculus Exam

BC-2: Solutions, Concepts, and Scoring Guidelines

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## BC 2

$t$	0	2	6	8	10	12
$y'(t)$	4	8	-2	3	-1	-5

A particle moves in the coordinate plane with position  $(x(t), y(t))$  at time  $t$ , where  $t$  is measured in seconds and  $x(t)$  and  $y(t)$  are twice-differentiable functions, both measured in meters.

For all times  $t$ , the  $x$ -coordinate of the particle's position has derivative

$$x'(t) = \frac{t}{\sqrt{t^2 + 25}}.$$

Selected values of  $y'(t)$ , the derivative of  $y(t)$ , over the interval  $0 \leq t \leq 12$  seconds are shown in the table.

The position of the particle at time  $t = 12$  is  $(x(12), y(12)) = (4, -3)$ .

(a) Using correct units, find the speed of the particle at time  $t = 6$ .

## Key Concepts

### Speed of a Particle

Suppose a particle moves in the plane so that its position at time  $t$  is given by the parametric equations  $x = f(t)$  and  $y = g(t)$ .

Consider the vector function  $\mathbf{r} = \langle f(t), g(t) \rangle$ .

$\mathbf{r}(t)$  is the position vector of the point  $P(f(t), g(t))$ .

If  $\mathbf{r}(t) = \langle f(t), g(t) \rangle$  then  $\mathbf{r}'(t) = \langle f'(t), g'(t) \rangle$

The **velocity vector**  $\mathbf{v}(t)$  is given by

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

The **speed** of the particle at time  $t$  is the magnitude of the velocity vector.

$$\text{speed} = |\mathbf{v}(t)| = |\mathbf{r}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

**Solution**

$$x'(t) = \frac{t}{\sqrt{t^2 + 25}} \Rightarrow x'(6) = \frac{6}{\sqrt{6^2 + 25}} = \frac{6}{\sqrt{61}}$$

Use expression for  $x'(t)$ .

$$y'(6) = -2$$

Read the value  $y'(6)$  from the table.

$$\text{speed} = \sqrt{[x'(6)]^2 + [y'(6)]^2}$$

Speed at  $t = 6$ 

$$= \sqrt{\left(\frac{6}{\sqrt{61}}\right)^2 + (-2)^2}$$

Use values for  $x'(6)$  and  $y'(6)$ 

$$= \sqrt{\frac{36}{61} + 4} = \sqrt{\frac{280}{61}} = 2\sqrt{\frac{70}{61}} \text{ m/s}$$

Simplify

## Scoring Guidelines

$$(a) x'(6) = \frac{6}{\sqrt{6^2 + 25}} = \frac{6}{\sqrt{61}} \quad \text{and} \quad y'(6) = -2$$

$$\begin{aligned} \text{speed} &= \sqrt{[x'(6)]^2 + [y'(6)]^2} \\ &= \sqrt{\left(\frac{6}{\sqrt{61}}\right)^2 + (-2)^2} \\ &= \sqrt{\frac{36}{61} + 4} = 2\sqrt{\frac{70}{61}} \text{ m/s} \end{aligned}$$

2:  $\begin{cases} 1 : \text{expression for speed} \\ 1 : \text{answer with units} \end{cases}$

## Scoring Notes

1: expression for speed

- Earned for an expression for speed applied to our problem.
- Examples:

- $\sqrt{(x'(t))^2 + (y'(t))^2}$  ? - ?

- $\sqrt{(x'(t))^2 + (y'(t))^2}$  at  $t = 6$  1 - ?

- $\sqrt{(x'(6))^2 + (y'(6))^2}$  1 - ?

1: answer with units

- Earned for the correct numerical answer and the correct units.

- 2.142 or 2.143 m/s

- Examples

- $\sqrt{(x'(6))^2 + (y'(6))^2} = 2.143$  m/s 1 - 1

- $\sqrt{\left(\frac{6}{\sqrt{61}}\right)^2 + (-2)^2}$  m/s 1 - 1

- (b) Find the exact value of  $x(4)$ , the  $x$ -coordinate of the position of the particle at time  $t = 4$ .

## Key Concepts

### Net Change

#### The Fundamental Theorem of Calculus

Suppose  $f$  is continuous on  $[a, b]$ .

(1) If  $g(x) = \int_a^x f(t) dt$ , then  $g'(x) = f(x)$ .

(2)  $\int_a^b f(x) dx = F(b) - F(a)$ , where  $F$  is any antiderivative of  $f$ , that is,  $F' = f$ .

### A Closer Look

1. If  $f$  is a positive function, then  $g(x)$  may be interpreted as the area under the graph of  $f$  from  $a$  to  $x$ .
2.  $g$  is an area-so-far function or accumulation function (net area-so-far function).
3.  $F'(x)$ : rate of change of  $y = F(x)$  with respect to  $x$ .  
 $F(b) - F(a)$ : the net change in  $y$ .

## Net Change Theorem

The definite integral of a rate of change,  $F'$ , is the net change in the original function  $F$ :

$$\int_a^b F'(x) dx = F(b) - F(a)$$

Rearrange the terms: alternate interpretation.

$$\underbrace{F(b)}_{\text{End amount}} = \underbrace{F(a)}_{\text{Start amount}} + \underbrace{\int_a^b F'(x) dx}_{\text{Net change}}$$

**Solution**

$$x(4) = x(12) - \int_4^{12} x'(t) dt$$

Net Change Theorem

$$= (4) - \int_4^{12} \frac{t}{\sqrt{t^2 + 25}} dt$$

Value for  $x(12)$ ; expression for  $x'(t)$ 

$$= (4) - \left[ \sqrt{t^2 + 25} \right]_4^{12}$$

 $u$ -substitution

$$= (4) - \left[ \sqrt{169} - \sqrt{41} \right] = \sqrt{41} - 9$$

FTC; simplify

## Scoring Guidelines

$$\begin{aligned} \text{(b) } x(4) &= x(12) - \int_4^{12} x'(t) dt \\ &= (4) - \int_4^{12} \frac{t}{\sqrt{t^2 + 25}} dt \\ &= (4) - \left[ \sqrt{t^2 + 25} \right]_4^{12} \\ &= (4) - [\sqrt{169} - \sqrt{41}] = \sqrt{41} - 9 \end{aligned}$$

$$3: \begin{cases} 1 : \text{integral} \\ 1 : \text{uses } x(12) \\ 1 : \text{answer} \end{cases}$$

## Scoring Notes

1: integral

- Must be our definite integral:  $\int_4^{12} x'(t) dt$
- A missing  $dt$  is assumed to appear to the right of the last term after each integral symbol and before any comparison ( $=, \leq, \geq, <, >$ ).

1: uses  $x(12)$

- Earned for combining  $x(12)$  with the value of the definite integral.

1: answer

- Final numerical answer does not need to be simplified, units not necessary.

Examples

$$x(4) = 4 - \int_4^{12} \frac{t}{\sqrt{t^2 + 25}} dt = -2.596 \text{ (or } -2.597) \qquad 1 - 1 - 1$$

$$- \int_4^{12} \frac{t}{\sqrt{t^2 + 25}} + 4 = -2.596$$

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